

# Nonlinear Terms of MHD Equations for Homogeneous Magnetized Shear Flow<sup>1</sup>

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**Abstract.** We have derived the full set of MHD equations for incompressible shear flow of a magnetized fluid and considered their solution in the wave-vector space. The linearized equations give the famous amplification of slow magnetosonic waves and describe the magnetorotational instability. The nonlinear terms in our analysis are responsible for the creation of turbulence and self-sustained spectral density of the MHD (Alfvén and pseudo-Alfvén) waves. Perspectives for numerical simulations of weak turbulence and calculation of the effective viscosity of accretion disks are shortly discussed in k-space.

**Keywords:** missing viscosity, accretion disks, shear flow, nonlinear MHD term

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## 1. INTRODUCTION

How the chaotic matter did get organized into compact astrophysical objects, such as stars, when the Universe was created? How did the Sun's rotation get slowed down (a central problem of cosmogony)? What is the physics behind quasars' shining? The answers of all those unresolved problems of contemporary physics go back to the problem of the effective viscosity of weakly magnetized plasmas in shear rotating flows. For half a century we have faced a kinetic problem—how to calculate an effective viscosity—a problem that is at the core of the machine for making stars. This longstanding problem has already been approached in so many ways—any proposal-writing astrophysicist has already published his/her view and the literature of analytical works and numerical simulations is overwhelming. Yet the problem is still unsolved and has not lost its attractiveness. Here we will give it a try, too.

## 2. MODEL AND MHD EQUATIONS

Our starting point are the conservation laws for energy and momentum for an incompressible fluid with mass density  $\rho$

$$\partial_t(\rho V_i) + \partial_k(\Pi_{ik}) = 0, \quad (1)$$

$$\frac{\partial}{\partial t} \left( \frac{\rho V^2}{2} + \rho \tilde{\epsilon} + \frac{B^2}{2\mu_0} \right) + \text{div} \mathbf{q} = 0, \quad (2)$$

$$\text{div} \mathbf{V} = 0, \quad \rho = \text{const}, \quad (3)$$

where we have for total stress tensor  $\Pi$  and heat flux  $\mathbf{q}$  respectively

$$\Pi_{ik} = \rho V_i V_k + P \delta_{ik} - \eta \left( \frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i} \right) - \frac{1}{\mu_0} \left( B_i B_k - \frac{1}{2} \delta_{ik} B^2 \right), \quad (4)$$

$$\mathbf{q} = \rho \left( \frac{V^2}{2} + \tilde{w} \right) \mathbf{V} - \mathbf{V} \cdot \boldsymbol{\sigma}' - \kappa \nabla T + \frac{1}{\mu_0} [\mathbf{B} \times (\mathbf{V} \times \mathbf{B})] - \frac{\epsilon_0 c^2 \rho}{\mu_0} (\mathbf{B} \times \text{rot} \mathbf{B}), \quad (5)$$

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where  $\tilde{\varepsilon}$  is the internal energy per unit mass,  $\tilde{w}$  is the enthalpy per unit mass,  $\sigma'_{ij} \equiv \eta(\partial_i V_k + \partial_k V_i)$  the viscous part of the stress tensor for an incompressible fluid,  $\eta$  is the viscosity,  $\kappa$  is the heat conductivity,  $T$  is the temperature,  $\rho$  is the Ohmic resistivity. The formulas are written in SI, for a transition to Gaussian system we substitute  $\mu_0 = 4\pi$  and  $\varepsilon_0 = 1/4\pi$ , i.e., expressions are written in an invariant form.

For the magnetic field's energy density rate of change we have

$$\begin{aligned} \frac{1}{2\mu_0} \frac{\partial}{\partial t} B^2 &= \frac{1}{\mu_0} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0} \mathbf{B} \cdot [\nabla \times (\mathbf{V} \times \mathbf{B}) - v_m \nabla \times (\nabla \times \mathbf{B})] \\ &= \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times (\mathbf{V} \times \mathbf{B}) - v_m \mathbf{B} \times (\nabla \times \mathbf{B})). \end{aligned} \quad (6)$$

We calculate the divergence of the total stress tensor Eq. (4), and using that

$$\partial_k \frac{1}{\mu_0} \left( B_i B_k - \frac{1}{2} B^2 \delta_{ik} \right) = \frac{1}{\mu_0} \left( \mathbf{B} \cdot \nabla \mathbf{B} - \frac{1}{2} \nabla B^2 \right) = \frac{1}{\mu_0} (\mathbf{B} \times \text{rot} \mathbf{B}) = \mathbf{j} \times \mathbf{B}, \quad (7)$$

we obtain the equation of motion for an incompressible plasma,

$$\rho \partial_t \mathbf{V} = -\mathbf{V} \cdot \nabla \mathbf{V} - \nabla P + \mathbf{j} \times \mathbf{B} + \eta \nabla^2 \mathbf{V}. \quad (8)$$

To close the set of equations we need to use Ampère's law,  $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ , and generalized Ohm's law,  $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{j}$ , and supplement them with Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \quad (9)$$

For the second set of MHD equations we have

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{V} \times \mathbf{B} - \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right). \quad (10)$$

The MHD equations for an incompressible fluid  $\rho = \text{const}$ , in homogeneous magnetic field  $\mathbf{B}_0$ , shear flow with rate  $A$ , and angular velocity  $\boldsymbol{\Omega} = A\omega \mathbf{e}_z = A\boldsymbol{\omega}$  are

$$\rho D_t \mathbf{V} = -\nabla P + \left( \mathbf{j} = \frac{\nabla \times \mathbf{B}}{\mu_0} \right) \times \mathbf{B} - 2\rho \boldsymbol{\Omega} \times \mathbf{V} + \rho v_k \nabla^2 \mathbf{V}, \quad (11)$$

$$D_t \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V} + v_m \nabla^2 \mathbf{B}, \quad \text{div} \mathbf{V} = 0, \quad \text{div} \mathbf{B} = 0, \quad (12)$$

where  $D_t \equiv \partial_t + \mathbf{V} \cdot \nabla$  is the substantial (convective) derivative,  $P$  is the pressure,  $\mathbf{j}$  is the current density and  $v_k$  is the kinematic viscosity, The magnetic diffusivity  $v_m = \varepsilon_0 c^2 \rho$  is expressed by the Ohmic resistance  $\rho$  and  $\varepsilon_0 = 1/\mu_0 c^2$ . In order to obtain a linear system of dimensionless MHD equations we use the following *Ansatz* for the velocity  $\mathbf{V}$ , the magnetic field  $\mathbf{B}$ , the wave vector  $\mathbf{Q}$ , and the pressure  $P$

$$\mathbf{V}(t, \mathbf{r}) = \mathbf{V}_{\text{shear}}(\mathbf{r}) + \mathbf{V}_{\text{wave}}(t, \mathbf{r}), \quad \mathbf{V}_{\text{wave}} = iV_A \sum_{\mathbf{Q}} \mathbf{v}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}}, \quad \mathbf{V}_{\text{shear}} = Ax \mathbf{e}_y, \quad (13)$$

$$\mathbf{B}(t, \mathbf{r}) = \mathbf{B}_0 + \mathbf{B}_{\text{wave}}(t, \mathbf{r}), \quad \mathbf{B}_{\text{wave}}(t, \mathbf{r}) = B_0 \sum_{\mathbf{Q}} \mathbf{b}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}}, \quad (14)$$

$$P(t, \mathbf{r}) = P_0 + P_{\text{wave}}(t, \mathbf{r}), \quad P_{\text{wave}}(t, \mathbf{r}) = \rho V_A^2 \sum_{\mathbf{Q}} \mathcal{P}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}}. \quad (15)$$

where the sums are actually integrals with respect to 3-dimensional Eulerian wave-vector space with independent coordinates  $Q_x, Q_y, Q_z$

$$\sum_{\mathbf{Q}} = \int \int \int_{-\infty, -\infty, -\infty}^{+\infty, +\infty, +\infty} \frac{dQ_x dQ_y dQ_z}{(2\pi)^3} = \int d^3 \left( \frac{\mathbf{Q}}{2\pi} \right) = \int \frac{d^3 Q}{(2\pi)^3}, \quad (16)$$

i.e., the sum is a short notation for Fourier integration with omitted differentials, integral limits and  $2\pi$  multipliers. For the static magnetic  $\mathbf{B}_0$  field with magnitude  $B_0 = \sqrt{B_{0y}^2 + B_{0z}^2}$  we suppose a vertical  $B_{0z}$  and an azimuthal  $B_{0y}$

components parameterized by an angle  $\theta$  and an unit vector  $\boldsymbol{\alpha}$ . We also assume that the Alfvén velocity  $V_A$  is much smaller than the sound speed  $c_s$

$$\mathbf{B}_0 = B_0 \boldsymbol{\alpha}, \quad \boldsymbol{\alpha} = (0, \alpha_y = \sin \theta, \alpha_z = \cos \theta), \quad \mathbf{V}_A = \frac{\mathbf{B}_0}{\sqrt{\mu_0 \rho}}, \quad V_A = \frac{B_0}{\sqrt{\mu_0 \rho}}. \quad (17)$$

In the equations above we used the dimensionless space-vector

$$\mathbf{X} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \equiv \frac{\mathbf{r}}{\Lambda} = \frac{A \mathbf{r}}{V_A}, \quad \Lambda \equiv \frac{V_A}{A}, \quad \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (18)$$

and its dimensionless wave-vector counterpart  $\mathbf{Q}$ . Here,  $\Lambda$  is the characteristic length of the system which we suppose to be much smaller than space inhomogeneities, e.g., the accretion disk thickness.

### 3. WAVE-VECTOR REPRESENTATION

#### 3.1. Linear terms

We have a space homogeneous physical system and indispensably its modes bear the character of plane waves. The purpose of the present section is to find the Fourier transformation of the all the terms in the MHD equations (13), (14) and (15).

Let us start, for example, with the pressure. According to Eq. (15) we have

$$\frac{\nabla P}{\rho} = \frac{iV_A^2}{\Lambda} \int P_{\mathbf{Q}}(\tau) \mathbf{Q} e^{i\mathbf{Q} \cdot \mathbf{X}} \frac{d^3 Q}{(2\pi)^3} = iAV_A \sum_{\mathbf{Q}} \mathbf{Q} P_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}} \quad (19)$$

$$\hat{\mathcal{F}}\left(\frac{\nabla P}{\rho}\right) \equiv \int e^{-i\mathbf{Q} \cdot \mathbf{X}} \frac{\nabla P}{\rho} d^3 X = iAV_A \mathbf{Q} P_{\mathbf{Q}}(\tau). \quad (20)$$

Analogously, according to Eq. (13) and Eq. (14), for the partial time derivatives we obtain

$$\partial_t \mathbf{V} = iAV_A \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} \partial_{\tau} \mathbf{v}_{\mathbf{Q}}(\tau), \quad \hat{\mathcal{F}}(\partial_t \mathbf{V}) = iAV_A \partial_{\tau} \mathbf{v}_{\mathbf{Q}}(\tau), \quad (21)$$

$$\partial_t \mathbf{B} = AB_0 \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} \partial_{\tau} \mathbf{b}_{\mathbf{Q}}(\tau), \quad \hat{\mathcal{F}}(\partial_t \mathbf{B}) = AB_0 \partial_{\tau} \mathbf{b}_{\mathbf{Q}}(\tau). \quad (22)$$

More complicated are the Fourier transformations of the expressions, containing the shear flow  $\mathbf{V}_{\text{shear}} = V_A X \mathbf{e}_y$  and the wave variables

$$\mathbf{V}_{\text{shear}} \cdot \nabla \mathbf{V}_{\text{wave}} = -AV_A \int e^{i\mathbf{Q} \cdot \mathbf{X}} X Q_y \mathbf{v}(\tau) \frac{d^3 Q}{(2\pi)^3}, \quad (23)$$

$$\mathbf{V}_{\text{shear}} \cdot \nabla \mathbf{B}_{\text{wave}} = iAB_0 \int e^{i\mathbf{Q} \cdot \mathbf{X}} X Q_y \mathbf{b}(\tau) \frac{d^3 Q}{(2\pi)^3}. \quad (24)$$

Let variables  $\mathbf{V}_{\text{wave}}$  or  $\mathbf{B}_{\text{wave}}$  be presented by their Fourier components  $\psi(\mathbf{X}) = \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} \psi_{\mathbf{Q}}$  and  $\psi_{\mathbf{Q}} = \hat{\mathcal{F}}(\psi(\mathbf{X}))$ . Our task is to derive the Fourier transformation  $\hat{\mathcal{F}}(\mathbf{X} \psi(\mathbf{X}))$ . Using that  $\mathbf{X} e^{i\mathbf{Q} \cdot \mathbf{X}} = -i\partial_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}}$  and the Gaussian theorem,  $\int_{\mathcal{V}} d^3 Q \partial_{\mathbf{Q}} = \oint_{\partial \mathcal{V}} d\mathbf{S}$  applied for the whole volume  $\mathcal{V}$  in wave-vector space and its boundary  $\partial \mathcal{V}$  we can make the partial integration

$$\mathbf{X} \psi(\mathbf{X}) = \mathbf{X} \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} \psi_{\mathbf{Q}} = -i \sum_{\mathbf{Q}} \left\{ (\partial_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}}) \psi_{\mathbf{Q}} = -e^{i\mathbf{Q} \cdot \mathbf{X}} \partial_{\mathbf{Q}} \psi_{\mathbf{Q}} + \partial_{\mathbf{Q}} \left[ e^{i\mathbf{Q} \cdot \mathbf{X}} \partial_{\mathbf{Q}} \psi_{\mathbf{Q}} \right] \right\} = \sum_{\mathbf{Q}} e^{i\mathbf{Q} \cdot \mathbf{X}} i \partial_{\mathbf{Q}} \psi_{\mathbf{Q}}, \quad (25)$$

in the limit

$$\lim_{Q \rightarrow \infty} (Q^3 \psi_{\mathbf{Q}}) = 0.$$

In such a way we derived the well-known in quantum mechanics operator representation  $\hat{\mathbf{X}} = i\partial_{\mathbf{Q}}$  and derived the Fourier transformation

$$\hat{\mathcal{F}}(\mathbf{X}\psi(\mathbf{X})) = i\partial_{\mathbf{Q}}\psi_{\mathbf{Q}}. \quad (26)$$

This expression is analogous to the Fourier transformation of the  $\nabla$ -operator

$$\hat{\mathcal{F}}(\nabla_{\mathbf{X}}) = i\mathbf{Q}, \quad (27)$$

and yields

$$\hat{\mathcal{F}}(X\mathbf{e}_y \cdot \nabla_{\mathbf{X}}) = -Q_y \partial_{Q_x}, \quad \hat{\mathcal{F}}(\mathbf{V}_{\text{shear}} \cdot \nabla) = -A Q_y \partial_{Q_x}, \quad \mathbf{V}_{\text{shear}} = V_A X \mathbf{e}_y. \quad (28)$$

Those relations give that

$$\hat{\mathcal{F}}\left[D_t^{\text{shear}} \equiv \partial_t + \mathbf{V}_{\text{shear}} \cdot \nabla = \partial_t + AX \partial_y\right] = A \left\{ D_{\tau}^{\text{shear}} \equiv \partial_{\tau} - Q_y \partial_{Q_x} = \partial_{\tau} + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}} \right\}. \quad (29)$$

In other words Fourier transformation of a linearized substantial derivative is again a linearized substantial derivative, but only in the wave-vector space. For this purpose we introduced the field of shear flow in the wave-vector space  $\mathbf{U}_{\text{shear}}(\mathbf{Q}) \equiv -Q_y \mathbf{e}_x$ ; confer this result with  $\mathbf{V}_{\text{shear}}/V_A = X \mathbf{e}_y$ . Returning back to the velocity and magnetic field we arrive at

$$\hat{\mathcal{F}}[(\partial_t + \mathbf{V}_{\text{shear}} \cdot \nabla) \mathbf{V}_{\text{wave}}] = iAV_A [\partial_{\tau} + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}}] \mathbf{v}_{\mathbf{Q}}, \quad (30)$$

$$\hat{\mathcal{F}}[(\partial_t + \mathbf{V}_{\text{shear}} \cdot \nabla) \mathbf{B}_{\text{wave}}] = AB_0 [\partial_{\tau} + \mathbf{U}_{\text{shear}}(\mathbf{Q}) \cdot \partial_{\mathbf{Q}}] \mathbf{b}_{\mathbf{Q}}. \quad (31)$$

For the derivation of these equations we used Eq. (28) and according to Eqs. (21) and (22)  $\hat{\mathcal{F}}(\partial_t) = A\partial_{\tau}$ .

Very simple is the Fourier transformation of the dissipative terms which is reduced to the properties of the Laplacian

$$\begin{aligned} v_k \nabla^2 \mathbf{V} &= -v_k iV_A \int \mathbf{v}_{\mathbf{Q}}(\tau) Q^2 e^{i\mathbf{Q} \cdot \mathbf{X}} \frac{d^3 Q}{(2\pi)^3} = -\frac{iV_A}{\Lambda} v_k \int \mathbf{v}_{\mathbf{Q}}(\tau) Q^2 e^{i\mathbf{Q} \cdot \mathbf{X}} \frac{d^3 Q}{(2\pi)^3} = -iAV_A v'_k \int Q^2 \mathbf{v}_{\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}} \frac{d^3 Q}{(2\pi)^3}, \\ v_m \nabla^2 \mathbf{B} &= -\frac{B_0}{\Lambda^2} v_m \int Q^2 \mathbf{b}_{\mathbf{Q}}(\tau) \frac{d^3 Q}{(2\pi)^3} = -AB_0 v'_m \int Q^2 \mathbf{b}_{\mathbf{Q}}(\tau) \frac{d^3 Q}{(2\pi)^3}, \\ \frac{1}{iAV_A} \hat{\mathcal{F}}(v \nabla^2 \mathbf{V}) &= -v'_{\text{kin}} Q^2 \mathbf{v}(\tau), \quad \frac{1}{AB_0} \hat{\mathcal{F}}(v_m \nabla^2 \mathbf{B}) = -v'_{\text{m}} Q^2 \mathbf{b}(\tau), \quad v'_k \equiv \frac{A}{V_A^2} v_k, \quad v'_m \equiv \frac{A}{V_A^2} v_m. \end{aligned} \quad (32)$$

Hereafter for all terms coming from the velocity equation (11) we will separate a factor  $iAV_A$  and for all terms from Eq. (12) we will separate a factor  $AB_0$ . Those factors will be common for the final equations in the wave-vector space.

For Coriolis force density per unit mass  $-2\mathbf{\Omega} \times \mathbf{V}$ , we derive

$$\begin{aligned} -2\mathbf{\Omega} \times \mathbf{V} &= -2A\omega(-V_y \mathbf{e}_x + V_x \mathbf{e}_y) = 2A\omega \begin{pmatrix} V_y \\ -V_x \\ 0 \end{pmatrix} = 2A\omega \begin{pmatrix} Ax + iV_A \int v_{y,\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}} \frac{d^3 Q}{(2\pi)^3} \\ -iV_A \int v_{x,\mathbf{Q}}(\tau) e^{i\mathbf{Q} \cdot \mathbf{X}} \frac{d^3 Q}{(2\pi)^3} \\ 0 \end{pmatrix}, \\ \hat{\mathcal{F}}(-2\mathbf{\Omega} \times \mathbf{V}) &= iAV_A 2\omega(v_{y,\mathbf{Q}}(\tau) \mathbf{e}_x - v_{x,\mathbf{Q}}(\tau) \mathbf{e}_y). \end{aligned} \quad (33)$$

The centrifugal term  $2\omega A^2 x$  is irrelevant for the wave amplitude equations.

Furthermore, we calculate Lorentz force per unit mass  $\mathbf{j} \times \mathbf{B}$ , and using  $B_0^2/\mu_0 = \rho V_A^2$  we have

$$\begin{aligned} \left( \left( \frac{\nabla \times \mathbf{B}}{\mu_0} \right) \times \frac{\mathbf{B}_0}{\rho} \right) &= \frac{iB_0}{\mu_0 \rho \Lambda} \int (\mathbf{Q} \times \mathbf{b}_{\mathbf{Q}}) \times (B_{0y} \mathbf{e}_y + B_{0z} \mathbf{e}_z) e^{i\mathbf{Q} \cdot \mathbf{X}} \frac{d^3 Q}{(2\pi)^3} = iAV_A \int [(\mathbf{Q} \times \mathbf{b}_{\mathbf{Q}}(\tau)) \times \boldsymbol{\alpha}] \frac{d^3 Q}{(2\pi)^3}, \\ \hat{\mathcal{F}}\left(\left(\frac{\nabla \times \mathbf{B}}{\mu_0}\right) \times \frac{\mathbf{B}_0}{\rho}\right) &= iAV_A [(\mathbf{Q} \times \mathbf{b}_{\mathbf{Q}}(\tau)) \times \boldsymbol{\alpha}]. \end{aligned} \quad (34)$$

We have also other two zero terms having no influence on the wave dynamics. From momentum equation (11) and the magnetic field equation (12) we have

$$\mathbf{V}_{\text{shear}} \cdot \nabla \mathbf{V}_{\text{shear}} = Ax \mathbf{e}_y \cdot \nabla Ax \mathbf{e}_y = A^2 x \mathbf{e}_y \cdot \mathbf{e}_x \mathbf{e}_y = 0, \quad (35)$$

$$\mathbf{B}_0 \cdot \nabla \mathbf{V}_{\text{shear}} = AB_0 (\alpha_y \mathbf{e}_y + \alpha_z \mathbf{e}_z) \cdot \mathbf{e}_x \mathbf{e}_y = 0. \quad (36)$$

For the last linear terms we have

$$\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}} = iAV_A \int \mathbf{v}_Q(\tau) \cdot \mathbf{e}_x \mathbf{e}_y e^{iQ \cdot X} \frac{d^3 Q}{(2\pi)^3}, \quad \hat{\mathcal{F}}(\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}}) = iAV_A v_{x,Q}(\tau) \mathbf{e}_y, \quad (37)$$

$$\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}} = AB_0 \int b_{x,Q}(\tau) \mathbf{e}_y e^{iQ \cdot X} \frac{d^3 Q}{(2\pi)^3}, \quad \hat{\mathcal{F}}(\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{shear}}) = AB_0 b_{x,Q}(\tau) \mathbf{e}_y, \quad (38)$$

$$\mathbf{B}_0 \cdot \nabla \mathbf{V}_{\text{wave}} = \frac{iV_A B_0}{\Lambda} \int (\boldsymbol{\alpha} \cdot iQ) \mathbf{v}_Q(\tau) e^{iQ \cdot X} \frac{d^3 Q}{(2\pi)^3}, \quad \hat{\mathcal{F}}(\mathbf{B}_0 \cdot \nabla \mathbf{V}_{\text{wave}}) = -AB_0 (\boldsymbol{\alpha} \cdot Q) \mathbf{v}_Q(\tau). \quad (39)$$

All linear terms are well-known from previous investigations of MHD waves in magnetized shear flows. In the next subsection we will derive the nonlinear terms describing the wave–wave interaction coming from the convective time derivative.

### 3.2. Nonlinear wave–wave interaction

In order to derive the nonlinear term in the momentum equation (11) we calculate  $\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}$  using  $\nabla X = \frac{A}{V_A} \mathbf{1} = \mathbf{1}/\Lambda$

$$\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} = \sum_{Q'} iV_A \mathbf{v}(\tau, Q') e^{iQ' \cdot X} \cdot \nabla \sum_{Q''} iV_A \mathbf{v}(\tau, Q'') e^{iQ'' \cdot X} = -iAV_A \sum_{Q'} \sum_{Q''} \mathbf{v}_{Q'} \cdot Q'' \mathbf{v}_{Q''} e^{i(Q'+Q'') \cdot X}. \quad (40)$$

For the sake of brevity in the last terms we will omit the time argument  $\tau$  and write the wave-vector argument  $\mathbf{Q}$  as index. We make a Fourier transformation and obtain

$$\begin{aligned} \hat{\mathcal{F}}(\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) &\equiv \int d^3 X e^{-iQ \cdot X} (\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) \\ &= -iAV_A \sum_{Q'} \sum_{Q''} \mathbf{v}_{Q'} \cdot Q'' \mathbf{v}_{Q''} \delta\left(\frac{Q'+Q''-Q}{2\pi}\right) = -iAV_A \sum_{Q'} \mathbf{v}_{Q-Q'} \mathbf{v}_{Q'} \cdot Q. \end{aligned} \quad (41)$$

The velocity and magnetic fields

$$\mathbf{V}_{\text{wave}}(\tau, \mathbf{X}) = iV_A \int e^{Q \cdot X} \mathbf{v}(\tau, Q) \frac{d^3 X}{(2\pi)^3}, \quad \mathbf{B}_{\text{wave}}(\tau, \mathbf{X}) = B_0 \int e^{Q \cdot X} \mathbf{b}(\tau, Q) \frac{d^3 X}{(2\pi)^3}, \quad (42)$$

have to be real, hence the Fourier components should be odd for the velocity and even for the magnetic field

$$\mathbf{v}_{-Q} = -\mathbf{v}_Q, \quad \mathbf{b}_{-Q} = \mathbf{b}_Q. \quad (43)$$

Analogously, for the Fourier component of the wave-wave interaction part of the Lorentz force  $\mathbf{j}_{\text{wave}} \times \mathbf{B}_{\text{wave}}$  we obtain

$$\hat{\mathcal{F}}\left(\frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}_{\text{wave}}) \times \mathbf{B}_{\text{wave}}\right) = \int d^3 X e^{-iQ \cdot X} \left(\frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}_{\text{wave}}) \times \mathbf{B}_{\text{wave}}\right) = iAV_A \sum_{Q'} (\mathbf{Q}' \times \mathbf{b}_{Q'}) \times \mathbf{b}_{Q-Q'}. \quad (44)$$

In such a way we derive the Fourier component of the nonlinear term of the momentum equation

$$\mathbf{N}_{v,Q} \equiv \frac{1}{iAV_A} \hat{\mathcal{F}}\left(-\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} + \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}_{\text{wave}}) \times \mathbf{B}_{\text{wave}}\right) \quad (45)$$

$$= \sum_{Q'} [\mathbf{v}_{Q-Q'} \mathbf{v}_{Q'} \cdot Q + (Q' \times \mathbf{b}_{Q'}) \times \mathbf{b}_{Q-Q'}]. \quad (46)$$

Similarly, for the other nonlinear terms  $\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}$  and  $\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}$  we have

$$\begin{aligned}\hat{\mathcal{F}}(\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) &= \int d^3X e^{-i\mathbf{Q} \cdot \mathbf{X}} (\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}}) \\ &= -\frac{B_0 V_A}{\Lambda} \sum_{\mathbf{Q}'} \sum_{\mathbf{Q}''} \mathbf{b}_{\mathbf{Q}'} \cdot \mathbf{Q}'' \mathbf{v}_{\mathbf{Q}''} \delta\left(\frac{\mathbf{Q}' + \mathbf{Q}'' - \mathbf{Q}}{2\pi}\right) = -AB_0 \sum_{\mathbf{Q}'} \mathbf{b}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{v}_{\mathbf{Q} - \mathbf{Q}'},\end{aligned}\quad (47)$$

$$\begin{aligned}\hat{\mathcal{F}}(\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}) &= \int d^3X e^{-i\mathbf{Q} \cdot \mathbf{X}} (\mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}) \\ &= -\frac{B_0 V_A}{\Lambda} \sum_{\mathbf{Q}'} \sum_{\mathbf{Q}''} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q}'' \mathbf{b}_{\mathbf{Q}''} \delta\left(\frac{\mathbf{Q}' + \mathbf{Q}'' - \mathbf{Q}}{2\pi}\right) = -AB_0 \sum_{\mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{b}_{\mathbf{Q} - \mathbf{Q}'},\end{aligned}\quad (48)$$

Those terms participate in the equation for the magnetic field. For their difference we have

$$\begin{aligned}\mathbf{N}_{b,\mathbf{Q}} &\equiv \frac{1}{AB_0} \hat{\mathcal{F}}(\mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} - \mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}}) \\ &= \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{b}_{\mathbf{Q} - \mathbf{Q}'} - \mathbf{b}_{\mathbf{Q}'} \cdot (\mathbf{Q} - \mathbf{Q}') \mathbf{v}_{\mathbf{Q} - \mathbf{Q}'}] \\ &= -\mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q} - \mathbf{Q}'})\end{aligned}\quad (49)$$

As the function in  $r$ -space

$$\text{rot}(\mathbf{V}_{\text{wave}} \times \mathbf{B}_{\text{wave}}) = \mathbf{B}_{\text{wave}} \cdot \nabla \mathbf{V}_{\text{wave}} - \mathbf{V}_{\text{wave}} \cdot \nabla \mathbf{B}_{\text{wave}} \quad (50)$$

has zero divergence

$$\text{div}[\text{rot}(\mathbf{V}_{\text{wave}} \times \mathbf{B}_{\text{wave}})] = 0 \quad (51)$$

its Fourier transform is transversal  $\mathbf{Q} \cdot \mathbf{N}_{b,\mathbf{Q}} = 0$  and automatically  $\mathbf{N}_{b,\mathbf{Q}}^\perp = \mathbf{N}_{b,\mathbf{Q}}$ .

In order to merge the so derived nonlinear terms in a next subsection we will rederive the linear terms in the Lagrangian wave-vector space.

#### 4. ELIMINATION OF PRESSURE IN THE FINAL MHD EQUATIONS

It is common in MHD to formally seek the limit of a particular expression for infinite sound speed  $c_s \rightarrow \infty$ . Due to the complexity of the problem this standard approach for consideration weak magnetic fields when  $V_A \ll c_s$  is inapplicable to our problem and we have to look for direct elimination of the pressure. After substituting Fourier transformations in Eqs. (11) and (12) we obtain

$$\begin{aligned}(\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_{\mathbf{Q}}) \mathbf{v}_{\mathbf{Q}} &= -v_{x,\mathbf{Q}} \mathbf{e}_y + \mathbf{Q} P_{\mathbf{Q}} + [(\mathbf{Q} \times \mathbf{b}_{\mathbf{Q}}) \times \boldsymbol{\alpha}] + 2\omega(v_{y,\mathbf{Q}} \mathbf{e}_x - v_{x,\mathbf{Q}} \mathbf{e}_y) - v'_k Q^2 \mathbf{v}_{\mathbf{Q}} \\ &\quad + \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q} - \mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}) \times \mathbf{b}_{\mathbf{Q} - \mathbf{Q}'}],\end{aligned}\quad (52)$$

$$(\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_{\mathbf{Q}}) \mathbf{b}_{\mathbf{Q}} = b_{x,\mathbf{Q}} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_{\mathbf{Q}} - v'_m Q^2 \mathbf{b}_{\mathbf{Q}} - \mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'} \times \mathbf{b}_{\mathbf{Q} - \mathbf{Q}'}) \quad (53)$$

For the sake of brevity we introduce

$$\mathcal{F}_{\mathbf{Q}} \equiv Q_y \frac{\partial \mathbf{v}_{\mathbf{Q}}}{\partial Q_x} - \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{v}_{\mathbf{Q}} + [(\mathbf{Q} \times \mathbf{b}) \times \boldsymbol{\alpha}] + 2\boldsymbol{\omega} \times \mathbf{v}_{\mathbf{Q}} + v'_k Q^2 \mathbf{v}_{\mathbf{Q}} + \sum_{\mathbf{Q}'} [\mathbf{v}_{\mathbf{Q} - \mathbf{Q}'} \mathbf{v}_{\mathbf{Q}'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}) \times \mathbf{b}_{\mathbf{Q} - \mathbf{Q}'}]. \quad (54)$$

Then the equation for the velocity can be rewritten as

$$\partial_\tau \mathbf{v}_{\mathbf{Q}} = P_{\mathbf{Q}} \mathbf{Q} + \mathcal{F}_{\mathbf{Q}}. \quad (55)$$

In order to express the pressure, we multiply both sides of this equation by  $\mathbf{Q}$

$$\partial_\tau (\mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}}) = Q^2 P_{\mathbf{Q}} + \mathbf{Q} \cdot \mathcal{F}_{\mathbf{Q}}. \quad (56)$$

The incompressibility condition  $\mathbf{Q} \cdot \mathbf{v}_Q = 0$  gives for the pressure the solution of the Poisson equation

$$\begin{aligned}\mathcal{P} = -\frac{\mathbf{Q} \cdot \mathcal{F}_Q}{Q^2} &= -\frac{1}{Q^2} \{ 2\mathbf{Q} \cdot \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{v}_Q + 2\boldsymbol{\omega} \times \mathbf{v}_Q + \mathbf{Q} \cdot [(\mathbf{Q} \times \mathbf{b}_Q) \times \boldsymbol{\alpha}] \} \\ &\quad - \frac{1}{Q^2} \sum_{Q'} \{ \mathbf{Q} \cdot \mathbf{v}_{Q-Q'} \mathbf{v}_{Q'} \cdot \mathbf{Q} + [(\mathbf{Q}' \times \mathbf{b}_{Q'}) \times \mathbf{b}_{Q-Q'}] \cdot \mathbf{Q} \},\end{aligned}\quad (57)$$

where we used the obvious vector relations

$$\mathbf{Q} \cdot \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{v}_Q = 2v_x Q_y, \quad \mathbf{Q} \cdot (\boldsymbol{\omega} \times \mathbf{v}_Q) = \omega (Q_y v_x - Q_x v_y), \quad (\mathbf{Q} \times \mathbf{b}_Q) \times \boldsymbol{\alpha} = (\mathbf{Q} \cdot \boldsymbol{\alpha}) (\mathbf{Q} \cdot \mathbf{b}) - \mathbf{Q}^2 (\mathbf{b} \cdot \boldsymbol{\alpha}). \quad (58)$$

This formula for the pressure we substitute in Eq. (55) which takes the form

$$\partial_\tau \mathbf{v}_Q = \mathcal{F}_Q^\perp = \mathcal{F}_Q - \frac{\mathbf{Q} \otimes \mathbf{Q}}{Q^2} \mathcal{F}_Q = \Pi^{\perp Q} \mathcal{F}_Q, \quad (59)$$

where

$$\Pi^{\perp Q} \equiv \mathbb{1} - \frac{\mathbf{Q} \otimes \mathbf{Q}}{Q^2} = \mathbb{1} - \mathbf{n} \otimes \mathbf{n}, \quad \mathbf{n} \equiv \frac{\mathbf{Q}}{Q} \quad (60)$$

is the projection operator which applies to the part of a vector, perpendicular to the wave-vector. In other words, the elimination of the pressure conserves the perpendicular part of the Fourier component of the force  $\mathcal{F}_Q$  in the used dimensionless variables. Equation (59) means that the velocity field remains orthogonal to the wave vector. If in the beginning  $\mathbf{Q} \cdot \mathbf{v}_Q(\tau_0) = 0$ , the evolution gives that  $\mathbf{Q} \cdot \mathbf{v}_Q(\tau) = 0$  for every  $\tau > \tau_0$ .

Using that for the velocity as applicable for every orthogonal vector

$$\Pi^{\perp} \partial_\tau \mathbf{v}_Q = \partial_\tau \mathbf{v}_Q, \quad \Pi^{\perp} \mathbf{v}_Q = \mathbf{v}_Q \quad (61)$$

we can rewrite Eq. (59) as

$$\Pi^{\perp} (\partial_\tau \mathbf{v}_Q - \mathcal{F}_Q) = 0. \quad (62)$$

In order to take into account the  $Q_y \partial \mathbf{v}_Q / \partial Q_x$  term in Eq. (54) we use the obvious relations

$$Q_y \frac{\partial}{\partial Q_x} \left( \mathbf{v}_Q \cdot \frac{\mathbf{Q} \otimes \mathbf{Q}}{Q^2} \right) = Q_y \frac{\partial \mathbf{v}_Q}{\partial Q_x} \cdot \frac{\mathbf{Q} \otimes \mathbf{Q}}{Q^2} + \frac{Q_y v_x Q}{Q^2} \mathbf{Q} = 0, \quad \mathbf{v}_Q \cdot \mathbf{Q} = 0. \quad (63)$$

Now we represent the projection of the advective term  $\mathbf{U}_{\text{shear}} \cdot \partial_Q \mathbf{v}_Q$  as

$$\Pi^{\perp Q} (\mathbf{U}_{\text{shear}} \cdot \partial_Q \mathbf{v}_Q) = -Q_y \frac{\partial \mathbf{v}_Q}{\partial Q_x} - n_y \mathbf{n} v_{x,Q} \quad (64)$$

and arrive at the momentum equation in the form where the projection operator exists explicitly only in the nonlinear term

$$\begin{aligned}(\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_Q) \mathbf{v}_Q(\tau) &= -v_{x,Q} \mathbf{e}_y + 2n_y \mathbf{n} v_{x,Q} + 2\omega \mathbf{n} (n_y v_{x,Q} - n_x v_{y,Q}) + 2\boldsymbol{\omega} \times \mathbf{v}_Q + (\boldsymbol{\alpha} \cdot \mathbf{Q}) \mathbf{b}_Q \\ &\quad - v'_k Q^2 \mathbf{v}_Q + \Pi^{\perp Q} \sum_{Q'} [\mathbf{v}_{Q-Q'} \mathbf{v}_{Q'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{Q'}) \times \mathbf{b}_{Q-Q'}],\end{aligned}\quad (65)$$

$$(\partial_\tau + \mathbf{U}_{\text{shear}} \cdot \partial_Q) \mathbf{b}_Q(\tau) = b_{x,Q} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_Q - v'_m Q^2 \mathbf{b}_Q - \mathbf{Q} \times \sum_{Q'} (\mathbf{v}_{Q'} \times \mathbf{b}_{Q-Q'}), \quad (66)$$

$$\mathbf{v}_Q(\tau_0) = \Pi^{\perp} \mathbf{v}_Q(\tau_0), \quad \mathbf{b}_Q(\tau_0) = \Pi^{\perp} \mathbf{b}_Q(\tau_0). \quad (67)$$

For numerical calculations the incompressibility conditions  $\mathbf{n} \cdot \mathbf{b}_Q = 0$  and  $\mathbf{n} \cdot \mathbf{v}_Q = 0$  can be used as a criterion for the error.

Using the relation

$$[\mathbf{U}_{\text{shear}} \cdot \partial_Q \mathbf{b}_Q(\tau)] \cdot \mathbf{Q} = Q_y b_{x,Q}, \quad (68)$$

one can easily check that the equation for the evolution of the magnetic field, Eq. (66), can also be presented as the evolution of its part, perpendicular to the wave-vector

$$\partial_\tau \mathbf{b}_Q + \Pi^\perp \mathbf{U}_{\text{shear}} \cdot \partial_Q \mathbf{b}_Q = \Pi^\perp \mathbf{e}_y \mathbf{e}_x \cdot \mathbf{b}_Q - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_Q - v'_m Q^2 \mathbf{b}_Q - \mathbf{Q} \times \sum_{Q'} (\mathbf{v}_{Q'} \times \mathbf{b}_{Q-Q'}). \quad (69)$$

Together with  $\mathbf{v}_Q = \Pi^\perp \mathbf{v}_Q$  this equation automatically gives  $\mathbf{b}_Q = \Pi^\perp \mathbf{b}_Q$  and  $\text{div} \mathbf{B} = 0$ .

In the matrix form the set of MHD equations reads as

$$\mathbf{D}_\tau \Psi = \mathbf{M} \Psi + \mathbf{N}, \quad (70)$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_{b,Q}^\perp \\ \mathbf{N}_{v,Q}^\perp \end{pmatrix} = \begin{pmatrix} -\mathbf{Q} \times \sum_{Q'} (\mathbf{v}_{Q'} \times \mathbf{b}_{Q-Q'}) \\ \Pi^\perp \mathbf{Q} \sum_{Q'} [\mathbf{v}_{Q-Q'} \mathbf{v}_{Q'} \cdot \mathbf{Q} + (\mathbf{Q}' \times \mathbf{b}_{Q'}) \times \mathbf{b}_{Q-Q'}] \end{pmatrix}, \quad (71)$$

and

$$\mathbf{M} = \left( \begin{array}{ccc|cc} -v'_m Q^2 & 0 & 0 & -Q_\alpha & 0 & 0 \\ 1 & -v'_m Q^2 & 0 & 0 & -Q_\alpha & 0 \\ 0 & 0 & -v'_m Q^2 & 0 & 0 & -Q_\alpha \\ \hline Q_\alpha & 0 & 0 & 2n_y n_x (\omega + 1) - v'_k Q^2 & -2n_x n_x \omega + 2\omega & 0 \\ 0 & Q_\alpha & 0 & 2n_y n_y (\omega + 1) - (2\omega + 1) & -2n_x n_y \omega - v'_k Q^2 & 0 \\ 0 & 0 & Q_\alpha & 2n_y n_z (\omega + 1) & -2n_x n_z \omega & -v'_k Q^2 \end{array} \right), \quad \Psi_Q = \begin{pmatrix} b_x \\ b_y \\ b_z \\ v_x \\ v_y \\ v_z \end{pmatrix} \quad (72)$$

with  $Q_\alpha \equiv \mathbf{Q} \cdot \boldsymbol{\alpha}$ .

The matrix can also be represented as

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{bb} & \mathbf{M}_{bv} \\ \mathbf{M}_{vb} & \mathbf{M}_{vv} \end{pmatrix}, \quad \Psi_Q = \begin{pmatrix} \mathbf{b} \\ \mathbf{v} \end{pmatrix}, \quad (73)$$

$$\mathbf{M}_{vv} = 2n_y \begin{pmatrix} n_x & 0 & 0 \\ n_y & 0 & 0 \\ n_z & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2\omega \begin{pmatrix} n_x n_y & (n_y^2 + n_z^2) & 0 \\ -(n_x^2 + n_z^2) & -n_x n_y & 0 \\ n_y n_z & -n_x n_z & 0 \end{pmatrix} - v'_k Q^2 \mathbf{1}, \quad (74)$$

$$\mathbf{M}_{vb} = Q_\alpha \mathbf{1}, \quad \mathbf{M}_{bv} = -Q_\alpha \mathbf{1}, \quad \mathbf{M}_{bb} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - v'_m Q^2 \mathbf{1}, \quad (75)$$

With the help of the matrices in this representation, in the next section we will make Lyapunov analysis of the linearized MHD equations.

## 5. LYAPUNOV ANALYSIS OF THE LINEARIZED SET OF EQUATIONS IN LAGRANGIAN VARIABLES

For small  $Q_y$  we may neglect the advective term  $\mathbf{U}_{\text{shear}} \cdot \partial_Q = -Q_y \partial_{Q_x}$ . Then the linearized MHD equations take the form

$$\mathbf{D}_\tau \Psi = \mathbf{M} \Psi, \quad (\mathbf{Q}, \mathbf{Q}) \cdot \Psi = 0. \quad (76)$$

To perform an instability analysis we make use of the exponential substitution  $\Psi = \exp(\lambda \tau) \psi$  which leads to an eigenvalue problem with transversality conditions

$$\mathbf{M}(\mathbf{Q}) \Psi_Q = \lambda \Psi_Q, \quad \mathbf{Q} \cdot \mathbf{b}_Q = 0 = \mathbf{Q} \cdot \mathbf{v}_Q; \quad (77)$$

to make it short we can further omit the index  $\mathbf{Q}$

$$(\mathbf{M} - \lambda \mathbf{1}) \Psi = 0, \quad \mathbf{Q} \cdot \mathbf{b} = 0 = \mathbf{Q} \cdot \mathbf{v}. \quad (78)$$

Should we substitute the incompressibility and transversality conditions

$$v_z = -\frac{Q_x v_x + Q_y v_y}{Q_z}, \quad b_z = -\frac{Q_x b_x + Q_y b_y}{Q_z}, \quad (79)$$

in the secular equation, we would end up with an overdetermined set of equations. To avoid it, we omit the equations which initially have  $\lambda b_z$  and  $\lambda v_z$  terms, i.e., the 3-rd and the 6-th rows in the secular equation. In such a way we derive a secular equation for a reduced matrix

$$(\tilde{\mathbf{M}} - \lambda \mathbf{1}) \psi = 0, \quad \psi = \begin{pmatrix} b_x \\ b_y \\ v_x \\ v_y \end{pmatrix}, \quad \tilde{\mathbf{M}} = \left( \begin{array}{cc|cc} -v'_m Q^2 & 0 & -Q_\alpha & 0 \\ 1 & -v'_m Q^2 & 0 & -Q_\alpha \\ \hline Q_\alpha & 0 & 2n_y n_x (\omega + 1) - v'_k Q^2 & -2n_x n_x \omega + 2\omega \\ 0 & Q_\alpha & 2n_y n_y (\omega + 1) - (2\omega + 1) & -2n_x n_y \omega - v'_k Q^2 \end{array} \right). \quad (80)$$

This secular equation

$$P_4(\lambda; \mathbf{Q}, v'_m, v'_k) \equiv \det(\tilde{\mathbf{M}} - \lambda \mathbf{1}) = 0 \quad (81)$$

has 4 eigenvalues and via a calculation of the eigenvectors, we can derive  $b_z$  and  $v_z$  according to the transversality conditions given by Eq. (79).

For an ideal fluid both  $v'_m$  and  $v'_k$  are equal to zero. Omitting the viscosity terms we have a relatively simple form for the secular equation

$$\begin{aligned} P_4(\lambda; \mathbf{Q}, v'_m = 0, v'_k = 0) = & \lambda^4 - 2n_y n_x \lambda^3 + \{[(4 - 8n_y^2)n_x^2 + 4 - 4n_y^2]\omega_c^2 + [(2 - 8n_y^2)n_x^2 - 4n_y^2 + 2]\omega_c + 2Q_\alpha^2\}\lambda^2 \\ & - 2Q_\alpha^2 n_y n_x \lambda + 2Q_\alpha^2 (n_x^2 + 1)\omega_c + Q_\alpha^4 = 0. \end{aligned} \quad (82)$$

As we pointed out these eigenvalues give only a WKB approximation for the dynamics of MHD variables  $\psi(\tau)$ . For the special case of  $Q_y = 0$ , which corresponds to an axial-symmetric motion, with a rotation along the  $z$ -axis, the secular equation gives directly the growth rates of the linearized MHD equations.

$$P_4(\lambda; Q_y = 0, v'_m = 0, v'_k = 0) = \lambda^4 + 2[Q_\alpha^2 + (1 + 2\omega_c)(n_x^2 + 1)\omega_c]\lambda^2 + 2Q_\alpha^2(n_x^2 + 1)\omega_c + Q_\alpha^4 = 0. \quad (83)$$

The most restricted case is for the wave-vectors parallel to the rotation axis  $\mathbf{Q} = Q \mathbf{e}_z$  when  $Q_\alpha = Q_z \cos \theta$

$$P_4(\lambda; Q_x = 0, Q_y = 0, v'_m = 0, v'_k = 0) = \lambda^4 + 2[Q_\alpha^2 + (1 + 2\omega_c)\omega_c]\lambda^2 + (Q_\alpha^2 + 2\omega_c)Q_\alpha^2 = 0. \quad (84)$$

This is perhaps the most cited bi-quadratic equation in the whole history of science because it describes the magnetorotational instability (MRI) discovered by Velikhov [4] in 1959. In the astrophysics, this equation was recognized and overexposed by many astrophysical grants 30 years later, see equation Eq. (111) of Ref. [2] and historical remarks therein. If we consider the special case of pure shear  $\omega_c = 0$  with  $Q_y = 0$  this dispersion equation gives the usual Alfvén waves

$$(\lambda^2 + Q_\alpha^2)^2 = 0, \quad \omega = |Q_\alpha|, \quad (85)$$

i.e., the rotation destabilizes the Alfvén waves. The polarization of the magnetic field and the velocity of the Alfvén waves are along the shear flow.

For pure axial magnetic field  $\mathbf{B} = B \mathbf{e}_z$ , i.e.,  $\boldsymbol{\alpha} = (0, 0, 1)$ , and  $Q_\alpha = Q_z$ . The matrix reduction is then given by simply erasing the  $z$ -components and taking into account only the  $x$ - and  $y$ -projections of the equations of motions

$$\tilde{\mathbf{M}}_{\text{MRI}} = \left( \begin{array}{cc|cc} 0 & 0 & -Q_z & 0 \\ 1 & 0 & 0 & -Q_z \\ \hline Q_z & 0 & 0 & 2\omega_c \\ 0 & Q_z & -(2\omega_c + 1) & 0 \end{array} \right), \quad \psi = \begin{pmatrix} b_x \\ b_y \\ v_x \\ v_y \end{pmatrix}. \quad (86)$$

The secular equation is equation (84) for the MRI with  $Q_\alpha = Q_z$ .

The projection method can be generalized in the general case if we introduce 2 unit vectors perpendicular to the wave-vector  $\mathbf{e}_Q = \mathbf{Q}/Q$

$$|2\rangle = \mathbf{e}_2 = \frac{\mathbf{e}_z \times \mathbf{e}_Q}{|\mathbf{e}_z \times \mathbf{e}_Q|} = \frac{1}{\sqrt{Q_x^2 + Q_y^2}} \begin{pmatrix} -Q_y \\ Q_x \\ 0 \end{pmatrix}, \quad (87)$$

$$|1\rangle = \mathbf{e}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_Q}{|\mathbf{e}_2 \times \mathbf{e}_Q|} = \frac{1}{\sqrt{Q_x^2 + Q_y^2} \sqrt{Q_x^2 + Q_y^2 + Q_z^2}} \begin{pmatrix} -Q_x Q_z \\ Q_y Q_z \\ -Q_y^2 - Q_x^2 \end{pmatrix}, \quad (88)$$

and also the corresponding bra-vectors

$$\langle 1 | = \frac{(-Q_x Q_z, Q_y Q_z, -Q_y^2 - Q_x^2)}{\sqrt{Q_x^2 + Q_y^2} \sqrt{Q_x^2 + Q_y^2 + Q_z^2}}, \quad (89)$$

$$\langle 2 | = \frac{(-Q_y, Q_x, 0)}{\sqrt{Q_x^2 + Q_y^2}}. \quad (90)$$

For the degenerated case of  $Q_x = 0 = Q_y$  we can regularize by choosing  $Q_x = \iota$  and  $Q_y = 0$ . Then the limit  $\iota \rightarrow 0$  gives the regularizations  $|1\rangle = \mathbf{e}_x$  and  $|2\rangle = \mathbf{e}_y$ . For all matrices  $M_{\alpha,\beta}$  where  $\alpha, \beta = b, v$  we calculate the matrix elements in the two-dimensional space

$$(\bar{M}_{\alpha,\beta})_{jj'} = \langle j | M_{\alpha,\beta} | j' \rangle, \quad \text{where } j, j' = 1, 2. \quad (91)$$

In such a way we obtain a reduced  $4 \times 4$  matrix

$$\bar{M} = \begin{pmatrix} \bar{M}_{bb} & | & \bar{M}_{bv} \\ \hline \bar{M}_{vb} & | & \bar{M}_{vv} \end{pmatrix} \quad (92)$$

whose eigenvectors are automatically perpendicular to  $\mathbf{Q}$ , simply because we have used the orthogonal to  $\mathbf{Q}$  space.

As a rule the linearized analysis is made in Lagrangian, moving, wave-vector space

$$d_\tau \mathbf{K}(\tau) = \mathbf{U}_{\text{shear}}(\mathbf{K}(\tau)), \quad (93)$$

with a time-dependent wave-vector

$$K_x = K_{x,0} - K_y(\tau - \tau_0), \quad K_y = \text{const}, \quad K_z = \text{const} \quad (94)$$

for each MHD wave.

In these coordinates for linearized waves the substantial time derivative  $D_\tau^{\text{shear}} = d_\tau$  is reduced to a usual time derivative and the separation of variables gives a set of ordinary independent equations for every MHD wave

$$d_\tau \Psi_{\mathbf{K}}(\tau) = M(\mathbf{K}(\tau)) \Psi_{\mathbf{K}}(\tau), \quad \mathbf{K}(\tau) \cdot \mathbf{v}_{\mathbf{K}}(\tau) = 0, \quad \mathbf{K}(\tau) \cdot \mathbf{b}_{\mathbf{K}}(\tau) = 0. \quad (95)$$

In this linearized case it is possible to exclude  $b_z$  and  $v_z$ . In such a way we arrive at a simple-for-programming set of 4 equations and 2 zero divergence conditions

$$d_\tau \Psi_{\mathbf{K}}(\tau) = \tilde{M}(\mathbf{K}(\tau)) \Psi_{\mathbf{K}}(\tau), \quad b_z = -(K_x(\tau)b_x + K_y b_y)/K_z, \quad v_z = -(K_x(\tau)v_x + K_y v_y)/K_z. \quad (96)$$

For small  $K_y$  one can apply the WKB approximation supposing exponential time dependence of the MHD variables  $\Psi(\tau) \propto \exp(\lambda \tau)$  and the wave amplitudes. In the WKB approximation the energy amplification between  $\tau = -\infty$  and  $\tau = +\infty$  is given by the eigenvalue  $\lambda$  with the maximal real part

$$G \approx \exp \left( 2 \int_{-\infty}^{\infty} d\tau \operatorname{Re} \lambda_{\max}(\mathbf{K}(\tau)) \right). \quad (97)$$

For the case of MRI with nonzero  $B_z$  the amplification factors are so giant that the linear analysis makes no sense because the nonlinear terms become rather important and we have a nonlinear saturation of the MRI. This saturation simulates strong turbulence for small wave-vectors, but definitely for large wave-vectors  $|K_y| \gg 1$  at  $\tau \rightarrow \infty$  we have a wave type turbulence with a given frequency.

We have to mention that the linearized case of pure shear is exactly integrable in terms of the Heun functions [1, 5]. Investigating numerically this case with  $\omega_c = 0$  and  $B_z = 0$  in his Ph.D. thesis [6] T. Hristov discovered in 1990 the amplification of slow magnetosonic waves (SMWs) by shear flows. Applied to the physics of accretion disks this amplification works even for purely azimuthal magnetic fields and gives a scenario for weak magnetic turbulence related to amplification of SMWs. We had to wait 30 years of incubation period, cf. [7], for the SMWs amplification to be recognized as an important issue for that astrophysical phenomenon. In the next section we will consider how to proceed with the solution to the MHD equations.

## 6. ENERGY DENSITY AND POWER DENSITY

Our first step is to calculate the energy of plane MHD waves with time-dependent amplitudes. Using that

$$\int e^{i\mathbf{Q}\cdot\mathbf{X}} d^3X = (2\pi)^3 \delta(\mathbf{Q}) \quad (98)$$

for the energy we obtain

$$\frac{1}{2} \int \left( \rho \mathbf{V}_{\text{wave}}^2 + \frac{1}{\mu_0} \mathbf{B}_{\text{wave}}^2 \right) d^3X = \rho V_A^2 \sum_{\mathbf{Q}} \varepsilon_{\mathbf{Q}}, \quad \varepsilon_{\mathbf{Q}} \equiv \frac{1}{2} (\mathbf{v}_{\mathbf{Q}}^2 + \mathbf{b}_{\mathbf{Q}}^2), \quad (99)$$

i.e., the energy density is

$$\rho V_A^2 \int \int \int \frac{1}{2} [\mathbf{v}_{\mathbf{Q}}^2(\tau) + \mathbf{b}_{\mathbf{Q}}^2(\tau)] \frac{dQ_x dQ_y dQ_z}{(2\pi)^3}. \quad (100)$$

Analogously, with the help of the viscous stress tensor  $\sigma'_{ik}$  we express the volume density of the wave heating

$$\begin{aligned} Q_{\text{kin}}^{\text{wave}} &= \int \sigma'_{ik} \partial_k V_i^{\text{wave}} d^3x = \frac{1}{2} \int \sigma'_{ik} (\partial_k V_i^{\text{wave}} + \partial_i V_k^{\text{wave}}) d^3x = \frac{\eta}{2} \int (\partial_k V_i^{\text{wave}} + \partial_i V_k^{\text{wave}})^2 d^3x \quad (101) \\ &= \frac{\eta V_A^2}{2\Lambda^2} \int \left( \sum_{\mathbf{Q}} Q_i v_k e^{i\mathbf{Q}\cdot\mathbf{X}} + \sum_{\mathbf{Q}'} Q'_k v_i e^{i\mathbf{Q}'\cdot\mathbf{X}} \right)^2 d^3x = \rho A V_A^2 v'_k \sum_{\mathbf{Q}} Q^2 v_{\mathbf{Q}}^2. \end{aligned}$$

Similarly for the Ohmic part of the energy dissipation rate we have

$$Q_{\Omega}^{\text{wave}} = \mathbf{j} \cdot \mathbf{E} = \frac{1}{\mu_0 \sigma_{\Omega}} (\text{rot} \mathbf{B}_{\text{wave}})^2 = \frac{B_0^2}{\mu_0^2 \sigma_{\Omega}} \left( \sum_{\mathbf{Q}} \nabla \times \mathbf{b}_{\mathbf{Q}} e^{i\mathbf{Q}\cdot\mathbf{X}} \right)^2 = \rho A V_A^2 v'_m \sum_{\mathbf{Q}} Q^2 b_{\mathbf{Q}}^2. \quad (102)$$

The dissipation rate of a laminar shear flow is given according to Newton's formula

$$Q_{\text{kin}}^{\text{shear}} = \frac{\eta}{2} \int (\partial_k V_i^{\text{shear}} + \partial_i V_k^{\text{shear}})^2 d^3x = \frac{\eta}{2} A^2 (\delta_{k,x} \delta_{i,y} + \delta_{i,x} \delta_{k,y})^2 = \eta A^2. \quad (103)$$

Now we can calculate the total energy dissipation  $Q_{\text{tot}} = Q_{\text{kin}}^{\text{shear}} + Q_{\text{kin}}^{\text{wave}} + Q_{\Omega}^{\text{wave}}$ , the viscosity and the effective viscosity  $\eta_{\text{eff}}$

$$\eta = \frac{Q_{\text{kin}}^{\text{shear}}}{A^2}, \quad \eta_{\text{eff}} = \rho v_{\text{eff}} = \frac{Q_{\text{tot}}}{A^2}. \quad (104)$$

In this way we can express the effective kinematic viscosity by the dimensionless Fourier components of the velocity and the magnetic field

$$v_{\text{eff}}(\tau) = v_k + v_k \sum_{\mathbf{Q}} Q^2 v_{\mathbf{Q}}^2(\tau) + v_m \sum_{\mathbf{Q}} Q^2 b_{\mathbf{Q}}^2(\tau). \quad (105)$$

For example, if we have static probability distribution functions for the velocity and the magnetic field, the enhancement factor of the effective viscosity is given by the time-averaged squares of the Fourier components for  $\tau \gg 1$

$$\frac{\eta_{\text{eff}}}{\eta} = 1 + \sum_{\mathbf{Q}} Q^2 \langle v_{\mathbf{Q}}^2 \rangle + \frac{v_m}{v_k} \sum_{\mathbf{Q}} Q^2 \langle b_{\mathbf{Q}}^2 \rangle; \quad (106)$$

this important parameter determines the work of the accretion discs as a machine for making of compact astrophysical objects. The most simple scenario is to have the solution to the static equations for the  $i$ -th iteration of  $\Psi_{\mathbf{Q}}$  and to

calculate the next  $(i+1)$ -th iteration

$$\begin{aligned} \partial_{\bar{\tau}} \mathbf{v}_{\mathbf{Q}}^{(i+1)} &= -Q_y \frac{\partial \mathbf{v}_{\mathbf{Q}}^{(i+1)}}{\partial Q_x} = -v_{x,\mathbf{Q}}^{(i+1)} \mathbf{e}_y + 2 \frac{Q_y v_{x,\mathbf{Q}}^{(i+1)}}{Q^2} \mathbf{Q} + 2\omega \left[ \mathbf{n} (n_y v_{x,\mathbf{Q}}^{(i+1)} - n_x v_{y,\mathbf{Q}}^{(i+1)}) + (v_{y,\mathbf{Q}}^{(i+1)} \mathbf{e}_x - v_{x,\mathbf{Q}}^{(i+1)} \mathbf{e}_y) \right] \\ &+ (\boldsymbol{\alpha} \cdot \mathbf{Q}) \mathbf{b}_{\mathbf{Q}}^{(i+1)} - v'_k Q^2 \mathbf{v}_{\mathbf{Q}}^{(i+1)} + \Pi^{\perp \mathbf{Q}} \sum_{\mathbf{Q}'} \left[ \mathbf{v}_{\mathbf{Q}'}^{(i)} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}^{(i)} \cdot \mathbf{Q}' + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}^{(i)}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(i)} \right], \end{aligned} \quad (107)$$

$$\partial_{\bar{\tau}} \mathbf{v}_{\mathbf{Q}}^{(i+1)} = -Q_y \frac{\partial \mathbf{b}_{\mathbf{Q}}^{(i+1)}}{\partial Q_x} = b_{x,\mathbf{Q}}^{(i+1)} \mathbf{e}_y - (\mathbf{Q} \cdot \boldsymbol{\alpha}) \mathbf{v}_{\mathbf{Q}}^{(i+1)} - v'_m Q^2 \mathbf{b}_{\mathbf{Q}}^{(i+1)} - \mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'}^{(i)} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(i)}), \quad (108)$$

$$\partial_{\bar{\tau}} \equiv -Q_y \frac{\partial}{\partial Q_x}, \bar{\tau} = -\frac{Q_x}{Q_y}, D_{\tau} = \partial_{\tau} + \partial_{\bar{\tau}}, \text{ for the independent variables } (\bar{\tau}, Q_y, Q_z), Q_x = -Q_y \bar{\tau}. \quad (109)$$

For cold protoplanetary disks the Ohmic resistivity of weakly ionized gas is very high and the effective viscosity is dominated in Eq. (106) by the  $v_m/v_k$  term, in other words, the viscosity of the protoplanetary disks is created by Ohmic dissipation. Completely opposite is the situation for the hot almost completely-ionized hydrogen plasma in quasars. The Ohmic resistivity is negligible and the effective viscosity is created by the Fourier components of the MHD waves  $\langle \mathbf{v}_{\mathbf{Q}}^2 \rangle$ . Only for small wave-vectors the MHD turbulence remains strong turbulence. At large wave-vectors we have weak wave turbulence with wave-vectors going to infinity. In the next section we will consider the stability conditions which have to be checked.

## 7. STABILITY

The linear Lyapunov analysis which we outlined in Sec. 5 gives the idea what we have to do when we obtain the static solution  $\Psi_{\mathbf{Q}}^{(0)} = (\mathbf{b}_{\mathbf{Q}}^{(0)}, \mathbf{v}_{\mathbf{Q}}^{(0)})$ . In order to investigate the stability of this static solution we have to consider a small time-dependent deviation from this solution  $\Psi_{\mathbf{Q}}^{(1)}(\tau) = (\mathbf{b}_{\mathbf{Q}}^{(1)}(\tau), \mathbf{v}_{\mathbf{Q}}^{(1)}(\tau))$ . In this case, neglecting the quadratic terms with respect to  $\Psi_{\mathbf{Q}}^{(1)}$ , we find that the nonlinear terms in the MHD equations are linear integral operators in the  $\mathbf{Q}$ -space

$$\hat{N}' \Psi_{\mathbf{Q}}^{(1)} = \begin{pmatrix} \sum_{\mathbf{Q}'} \left[ \mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}'}^{(1)} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}^{(0)} + \mathbf{Q} \cdot \mathbf{v}_{\mathbf{Q}'}^{(0)} \otimes \mathbf{v}_{\mathbf{Q}-\mathbf{Q}'}^{(1)} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}^{(0)}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(1)} + (\mathbf{Q}' \times \mathbf{b}_{\mathbf{Q}'}^{(1)}) \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(0)} \right] \\ - \mathbf{Q} \times \sum_{\mathbf{Q}'} (\mathbf{v}_{\mathbf{Q}'}^{(0)} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(1)} + \mathbf{v}_{\mathbf{Q}'}^{(1)} \times \mathbf{b}_{\mathbf{Q}-\mathbf{Q}'}^{(0)}) \end{pmatrix}. \quad (110)$$

We obtain new terms in the eigenvalue problem which finally is reduced to the problem of obtaining the maximal eigenvalue of an integral equation in which the coefficients are solutions to the static MHD equations. Now let us analyze the perspectives.

## 8. PERSPECTIVE

The open question of the missing viscosity in accretion disks is a longstanding problem in physics. In this work we have arrived at a complete set of numerically solvable ordinary differential equations which would help in finding a reasonable explanation of the aforementioned problem.

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